

# PRINCIPAL BASIS IN CARTAN SUBALGEBRA

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**ABSTRACT.** Let  $\mathfrak{g}$  be a simple complex Lie algebra and  $\mathfrak{h}$  a Cartan subalgebra. In this article we explain how to obtain the principal basis of  $\mathfrak{h}$  starting from a set of generators  $\{p_1, \dots, p_r\}, r = \text{rank}(\mathfrak{g})$ , of the invariants polynomials  $S(\mathfrak{g}^*)^{\mathfrak{g}}$ . For each invariant polynomial  $p$ , we define a  $G$ -equivariant map  $Dp$  from  $\mathfrak{g}$  to  $\mathfrak{g}$ . We show that the Gram-Schmidt orthogonalization of the elements  $\{Dp_1(\rho^\vee), \dots, Dp_r(\rho^\vee)\}$  gives the principal basis of  $\mathfrak{h}$ . Similarly the orthogonalization of the elements  $\{Dp_1(\rho), \dots, Dp_r(\rho)\}$  produces the principal basis of the Cartan subalgebra of  $\mathfrak{g}^\vee$ , the Langlands dual of  $\mathfrak{g}$ .

## CONTENTS

1. Introduction	1
2. Dual root system	3
3. Decomposition under action of the principal three dimensional subalgebra, and the principal basis of the Cartan subalgebras	5
4. How to obtain the principal basis of the Cartan subalgebra using the generators of the invariants polynomials	7
5. The $so(2l, \mathbb{C})$ case ( $l$ even)	11
References	14

## 1. INTRODUCTION

Let  $\mathfrak{g}$  be a simple Lie algebras over  $\mathbb{C}$ , let  $\mathfrak{h}$  be a choice of the Cartan subalgebra, and let  $\{\alpha_1, \dots, \alpha_r\}$  be a choice for the set of simple roots.. We consider the principal  $sl_2$ -triple,  $\mathfrak{s}_0 = \langle e_0, f_0, h_0 \rangle_{\mathbb{C}}$ , given by (Lemma 5.2 of [Kos])

the semi simple element :  $h_0 = 2\rho^\vee$  =the sum of positive coroots  
the positive nilpotent elements :  $e_0 = e_1 + \dots + e_r$  =the sum of positive nilpotent element corresponding to simple roots  
the negative nilpotent element :  $f_0$  such that the well know commutation relation are satisfied, i.e.  $[h_0, e_0] = 2e_0, [h_0, f_0] = -2f_0, [e_0, f_0] = h_0$ .

The restriction of the adjoint action of  $\mathfrak{g}$  to  $\mathfrak{s}_0$  gives a representation of  $\mathfrak{s}_0$  on  $\mathfrak{g}$ . Let

$$(1) \quad \mathfrak{g} = V_1 \oplus \dots \oplus V_r$$

be the decomposition of this representation into irreducible  $\mathfrak{so}_0$ -modules. There is exactly  $r = \text{rank}(\mathfrak{g})$  irreducible submodules and their dimensions are  $\dim(V_i) = 2k_i + 1$ , where the  $\{k_i\}_{i=1}^n$  are the exponents of  $\mathfrak{g}$  ([Kos], Section 5). Without loss of generality, we can suppose  $1 = k_1 \leq k_2 \leq \dots \leq k_r$ . Each of the modules  $V_i$  has a one dimensional intersection with  $\mathfrak{h}$ , i.e. for each  $i$  there exists  $h_i$  such that  $\langle h_i \rangle_{\mathbb{C}} = V_i \cap \mathfrak{h}$ . By definition ([Baz], Section 7.2) these elements form the principal basis of  $\mathfrak{h}$ ,

$$\text{principal basis} = \{h_1, \dots, h_r\}.$$

The principal basis was introduced by Kostant to solve a question about Clifford algebra (see remark 4.9).

If all exponents are distinct, the principal basis is uniquely defined (up to constant multiples). This is the case for all simple Lie algebras except for  $\mathfrak{so}(2l, \mathbb{C})$  with  $l$  even. In this case there are exactly two exponents with the value  $l - 1$ . Then in the decomposition (1) there are two modules of the same dimension. Hence the definition of the principal basis has to be refined in this case (see §5).

Let  $\{\tilde{p}_1, \dots, \tilde{p}_r\}$  be a choice of homogeneous generators in the ring of invariant polynomials  $S(\mathfrak{g}^*)^{\mathfrak{g}}$ . We suppose  $\deg(p_1) \leq \dots \leq \deg(p_r)$ . Let  $\{e_i^*\}_i$  be an orthonormal basis of  $\mathfrak{g}^*$  with respect to a choice of an invariant bilinear form  $B$ . We define

$$d\tilde{p}_i(x) = \sum_j \frac{\partial \tilde{p}_i}{\partial e_j^*}(x) \otimes e_j^* \quad x \in \mathfrak{g}$$

as a  $G$ -equivariant map  $d\tilde{p}_i : \mathfrak{g} \rightarrow \mathfrak{g}^*$ , and we consider the set

$$\{d\tilde{p}_1(\rho^\vee), \dots, d\tilde{p}_r(\rho^\vee)\}$$

of elements of  $\mathfrak{g}^*$ . Using  $B$  we can identify  $\mathfrak{g}$  with  $\mathfrak{g}^*$  and we can consider these elements as the elements in  $\mathfrak{g}$ . Our first result is (Theorem 4.3)

- (a) The elements  $\{d\tilde{p}_1(\rho^\vee), \dots, d\tilde{p}_r(\rho^\vee)\}$  form a basis of  $\mathfrak{h}$ .
- (b) After orthogonalization we obtain the principal basis of  $\mathfrak{h} \subset \mathfrak{g}$ .

Let  $\rho$  be the half sum of positive roots. Using  $B$  to identify  $\mathfrak{g}$  with  $\mathfrak{g}^*$  we construct the set

$$\{d\tilde{p}_1(\rho), \dots, d\tilde{p}_r(\rho)\}$$

of elements of  $\mathfrak{g}^*$ . Our second result is (Theorem 4.8)

- (a) The elements  $\{d\tilde{p}_1(\rho), \dots, d\tilde{p}_r(\rho)\}$  form a basis of  $\mathfrak{h}^\vee$ , the Cartan subalgebra of  $\mathfrak{g}^\vee$ , the Langlands dual of  $\mathfrak{g}$ .
- (b) After orthogonalization we obtain the principal basis of  $\mathfrak{h}^\vee \subset \mathfrak{g}^\vee$ .

In §2 we discuss the relation between roots and coroots and between the rings  $S(\mathfrak{g}^*)^{\mathfrak{g}}$ ,  $S(\mathfrak{h}^*)^{\mathcal{W}}$ ,  $S((\mathfrak{g}^\vee)^*)^{\mathfrak{g}^\vee}$  and  $S((\mathfrak{h}^\vee)^*)^{\mathcal{W}^\vee}$ .

In §3 we define the principal three dimensional subalgebra and the principal basis.

§4 contains the proof of the first two results, Theorem 4.3 and 4.8.

In §5 we study in detail the case of  $so(2l, \mathbb{C})$  ( $l$  even). This Lie algebra has two exponents with the same value, and the principal basis defined by the decomposition (1) is not unique. Let  $\sigma$  be the automorphism of order two of the Dinkin diagram of type  $D_l$  ( $l$  even). If we require that the vectors of the principal basis be eigenvectors of  $\sigma$ , then the principal basis is unique (up to constant multiple) and orthogonal (Theorem 5.1).

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## 2. DUAL ROOT SYSTEM

Throughout this section  $\mathfrak{g}$  denotes a simple Lie algebra over the field of complex numbers,  $\mathfrak{h} \subset \mathfrak{g}$  a Cartan subalgebra,  $\mathcal{R}$  the root system and  $B$  a non-degenerate invariant bilinear form on  $\mathfrak{g}$ , e.g. the Killing form or the canonical form.

The bilinear form  $B$  induces an isomorphism of vector spaces between  $\mathfrak{g}$  and  $\mathfrak{g}^*$ ,

$$B^\natural : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}^*,$$

and its inverse

$$B^\flat : \mathfrak{g}^* \xrightarrow{\cong} \mathfrak{g}.$$

They are given by  $B^\natural(x) = B(x, \cdot)$  and  $B^\flat(\alpha) = x_\alpha$ , where  $x_\alpha$  is the unique element of  $\mathfrak{g}$  such that  $B(x_\alpha, \cdot) = \alpha$ . As the restriction of  $B$  to the Cartan subalgebra  $\mathfrak{h}$  is non degenerate, the restriction of  $B^\natural$  and  $B^\flat$  to the Cartan subalgebra are isomorphisms. Moreover  $B$  induces a non degenerate bilinear form on  $\mathfrak{h}^*$  which will be denoted again by  $B$  and it is given for all  $\alpha, \beta \in \mathfrak{h}^*$  by

$$B(\alpha, \beta) = B(B^\flat(\alpha), B^\flat(\beta)).$$

To each root  $\alpha \in \mathfrak{h}^*$  we associate a coroot  $\alpha^\vee = \gamma(\alpha) \in \mathfrak{h}$ ,

$$(2) \quad \mathfrak{h}^* \ni \alpha \xrightarrow{\gamma} \alpha^\vee = \frac{2}{B(\alpha, \alpha)} B^\flat(\alpha) \in \mathfrak{h}.$$

The dual roots play the role of the roots in the dual root system  $\mathcal{R}^\vee$  of  $\mathfrak{g}$ . The map  $\gamma$  defines an isomorphism in the sense that  $\gamma : \mathcal{R} \xrightarrow{\cong} \mathcal{R}^\vee$ . Note that  $\mathcal{R}^\vee \subset \mathfrak{h}$ . The map  $\gamma$  is canonical, i.e. it is independent of the bilinear form  $B$ , but it does not extend to a vector space isomorphism between  $\mathfrak{h}^*$  and  $\mathfrak{h}$ .

Now we will recall the definition of the Langlands dual  $\mathfrak{g}^\vee$  of  $\mathfrak{g}$ .

*The Langlands dual  $\mathfrak{g}^\vee$  is the Lie algebra whose root system is  $\mathcal{R}^\vee$ .*

This definition is equivalent to :

*The Cartan matrix of  $\mathfrak{g}^\vee$  is the transpose of the Cartan matrix of  $\mathfrak{g}$ .*

or

*The Dynkin diagram of  $\mathfrak{g}^\vee$  is that of  $\mathfrak{g}$  with arrow reversed.*

The Langlands dual of the Langlands dual of  $\mathfrak{g}$  is  $\mathfrak{g}$ . For the root systems of simple Lie algebras we have the  $X_n^\vee = X_n$  for  $X = A, D, E, F, G$  and  $B_n^\vee = C_n$ .

As  $\mathcal{R}$  spans  $\mathfrak{h}^*$ , dual to the Cartan subalgebra of  $\mathfrak{g}$ , the dual root system  $\mathcal{R}^\vee$  spans  $(\mathfrak{h}^\vee)^*$ , dual to the Cartan subalgebra of  $\mathfrak{g}^\vee$ . We have the following identifications

$$\mathfrak{h}^\vee = \mathfrak{h}^* \xrightarrow[\cong]{\text{by } B} \mathfrak{h} = (\mathfrak{h}^\vee)^*.$$

Note that the left and right identifications are canonical, but the middle one depends on  $B$ .

Let  $\mathcal{W}$  and  $\mathcal{W}^\vee$  be the Weyl groups of  $\mathfrak{g}$  and  $\mathfrak{g}^\vee$  respectively. Let  $\sigma_\alpha \in \mathcal{W}$  and  $\sigma_{\alpha^\vee} \in \mathcal{W}^\vee$  be reflections by root  $\alpha$  and coroot  $\alpha^\vee$  respectively. We have that  $B^b$  intertwine them, i.e.

$$B^b \circ \sigma_\alpha = \sigma_{\alpha^\vee} \circ B^b.$$

The isomorphism  $B^b$  extends to a graded algebra isomorphism

$$B^b : S(\mathfrak{h}^*) \xrightarrow{\cong} S((\mathfrak{h}^\vee)^*),$$

and because it intertwines the action of the Weyl groups, its restriction to the Weyl group invariants is a graded algebra isomorphism

$$B^b : S(\mathfrak{h}^*)^{\mathcal{W}} \xrightarrow{\cong} S((\mathfrak{h}^\vee)^*)^{\mathcal{W}^\vee}.$$

Using the famous result [Che] of Chevalley we have that  $S(\mathfrak{h}^*)^{\mathcal{W}}$  and  $S((\mathfrak{h}^\vee)^*)^{\mathcal{W}^\vee}$  are generated by  $r = \text{rank}(\mathfrak{g})$  homogeneous linearly independent polynomials. Let  $\{p_1, \dots, p_r\}$  be a choice of generators of  $S(\mathfrak{h}^*)^{\mathcal{W}}$ . Then  $\{p_1^b = B^b(p_1), \dots, p_r^b = B^b(p_r)\}$  are generators of  $S((\mathfrak{h}^\vee)^*)^{\mathcal{W}^\vee}$ , i.e.

$$(3) \quad S(\mathfrak{h}^*)^{\mathcal{W}} = \mathbb{C}[p_1, \dots, p_r] \xrightarrow{\cong} S((\mathfrak{h}^\vee)^*)^{\mathcal{W}^\vee} = \mathbb{C}[p_1^b, \dots, p_r^b],$$

Moreover the  $p_i$ 's (and also the  $p_i^b$ 's) are homogeneous of degree  $k_i + 1$ , where the integers  $\{k_i\}_{i=1}^r$  are the exponents of  $\mathfrak{g}$ . Note that  $\mathfrak{g}$  and  $\mathfrak{g}^\vee$  have the same exponents. For simple Lie algebra we can choose these generators so that  $1 = k_1 \leq k_2 \leq \dots \leq k_r$  and the  $p_1$  is given by the restriction of  $B$  to  $\mathfrak{h}^*$ .

*Remark 2.1.* Using the Tables of [Bou], we notice that for a simple Lie algebra its exponents are all different, except for simple Lie algebras with roots system of type  $D_l$  with  $l$  even and greater or equal to 4.

Let  $\text{Chev} : S(\mathfrak{g}^*) \rightarrow S(\mathfrak{h}^*)$  be the Chevalley projection. Its restriction to the invariants induces a graded algebra isomorphism (see [Dix], Chapter 7, Section 3), i.e  $\text{Chev} : S(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{\cong} S(\mathfrak{h}^*)^{\mathcal{W}}$ . This isomorphism implies that  $S(\mathfrak{g}^*)^{\mathfrak{g}}$  and  $S((\mathfrak{g}^{\vee})^*)^{\mathfrak{g}^{\vee}}$  are generated by  $r$  linearly independent polynomials. We obtain the following commutative diagram of graded algebra isomorphisms :

$$(4) \quad \begin{array}{ccc} S(\mathfrak{g}^*)^{\mathfrak{g}} = \mathbb{C}[\tilde{p}_1, \dots, \tilde{p}_r] & \xrightarrow{\cong} & S((\mathfrak{g}^{\vee})^*)^{\mathfrak{g}^{\vee}} = \mathbb{C}[\hat{p}_1, \dots, \hat{p}_r] \\ \text{Chev} \downarrow & & \downarrow \text{Chev} \\ S(\mathfrak{h}^*)^{\mathcal{W}} = \mathbb{C}[p_1, \dots, p_r] & \xrightarrow{B^{\flat}} & S((\mathfrak{h}^{\vee})^*)^{\mathcal{W}^{\vee}} = \mathbb{C}[p_1^{\flat}, \dots, p_r^{\flat}] \end{array}$$

Moreover we can choose the generators so that :

- (a)  $\text{Chev}(\tilde{p}_i) = p_i$  and  $\text{Chev}(\hat{p}_i) = p_i^{\flat}$ ,
- (b) the upper horizontal isomorphism sends the generators  $\tilde{p}_i$  to the generators  $\hat{p}_i$ .

Note that  $\tilde{p}_1$  (resp.  $\hat{p}_1$ ) are defined up to a constant by an invariant bilinear form of  $\mathfrak{g}$  (resp.  $\mathfrak{g}^*$ ).

### 3. DECOMPOSITION UNDER ACTION OF THE PRINCIPAL THREE DIMENSIONAL SUBALGEBRA, AND THE PRINCIPAL BASIS OF THE CARTAN SUBALGEBRAS

Let  $\mathfrak{s} = \langle e, f, h \rangle_{\mathbb{C}}$  be a  $sl_2$ -triple,<sup>1</sup> of a simple complex Lie algebra  $\mathfrak{g}$ . The restriction of the adjoint representation of  $\mathfrak{g}$  to  $\mathfrak{s}$  give a representation of  $\mathfrak{s}$  on  $\mathfrak{g}$ , i.e.

$$\mathfrak{s} \ni x \rightarrow \text{ad}_x \in \text{End}(\mathfrak{g}).$$

is a Lie algebra homomorphism. Note that  $sl_2$ -triple exist, indeed every nilpotent elements of  $\mathfrak{g}$  can by a Morosov's Theorem embedded in a TDS (see Section 3 of [Kos]). We consider the decomposition of  $\mathfrak{g}$  into a direct sum of the irreducible  $\mathfrak{s}$ -modules

$$(5) \quad \mathfrak{g} = V_1 \oplus V_2 \oplus \dots \oplus V_n.$$

Without loss of generality we suppose that  $\dim(V_1) \leq \dim(V_2) \leq \dots \leq \dim(V_n)$ . In general the number  $n$  of irreducible  $\mathfrak{s}$ -modules is greater than the rank of  $\mathfrak{g}$  (see [Kos], Section 5), and if its equal to  $r$  then  $\mathfrak{s}$  will be call principal  $sl_2$ -triple (Theorem 5.2 in [Kos]).

We define a particular principal  $sl_2$ -triple,  $\mathfrak{s}_0 = \langle h_0, e_0, f_0 \rangle_{\mathbb{C}}$ , with

$$(6) \quad h_0 = 2\rho^{\vee}, \quad e_0 = \sum_i e_i \quad \text{and} \quad f_0 = \sum_i c_i f_i,$$

where  $\rho^{\vee}$  is the half sum of positive coroots,  $e_i$  (rep.  $f_i$ ) are the root vectors corresponding to the simple roots  $\alpha_i$  (resp.  $-\alpha_i$ ) such that  $B(e_i, f_i) = 1$  and the  $c_i$  are given by the relation  $\sum_i c_i B^{\flat}(\alpha_i) = 2\rho^{\vee}$ . Moreover each principal  $sl_2$ -triple is conjugate to (6), and if we require the semisimple element to be in  $\mathfrak{h}$ , then the principal  $sl_2$ -triple is given by (6). It is of course possible

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<sup>1</sup>The commutation relations are  $[h, e] = 2e$ ,  $[h, f] = -2f$  and  $[e, f] = h$ .

to multiply each  $e_i$  by a scalar and then divide  $f_i$  by the same scalar.

Using  $\mathfrak{s}_0$  in the decomposition (5) we have that  $n = \text{rank}(\mathfrak{g})$  and each  $V_i$  is an irreducible  $\mathfrak{s}_0$ -module. Moreover  $\dim(V_i) = 2k_i + 1$ , where the  $\{k_i\}_{i=1}^r$  are the exponents of  $\mathfrak{g}$ . This facts implies that for all  $i$  the intersection of  $V_i$  and  $\mathfrak{h}$  is one dimensional, i.e.

$$\text{for all } i, \text{ there exist } h_i \in \mathfrak{h} \text{ such that } \langle h_i \rangle_{\mathbb{C}} = V_i \cap \mathfrak{h}.$$

These elements form a basis of  $\mathfrak{h}$ . This basis is called the principal basis of  $\mathfrak{h}$  ([Baz], Section 7),

$$(7) \quad \text{principal basis of } \mathfrak{h} = \{h_1, \dots, h_r\}.$$

For  $k_i \neq k_j$  we have that  $h_i$  is orthogonal to  $h_j$  relatively to  $B$ . Indeed, we have that  $ad_{e_0} ad_{f_0}(h_i) = \frac{k_i(k_i+1)}{2} h_i$ , then using the invariance of  $B$  we conclude that for  $k_i \neq k_j$ ,  $h_i$  is orthogonal to  $h_j$ . Hence if all exponents are distinct, up to a constant, the principal basis is unique. In fact a similar argument shows that the decomposition (5) under  $\mathfrak{s}_0$  is an orthogonal decomposition. The only cases when there are two equal exponent occur for the Lie algebras with the root system of type  $D_l$  with  $l$  even, i.e. for  $so(2l, \mathbb{C})$ . We study this case in §5.

*Remark 3.1.* As vector space  $V_1 = \mathfrak{s}_0$ , then we have (up to a constant)  $h_1 = \rho^\vee$ .

*Remark 3.2.* The elements  $\{h_i\}_{i=1}^r$  of the principal basis are characterize by

- (a)  $h_i \in \text{Ker}(ad^{k_i+1}(e_0))$ ,
- (b)  $h_i \notin \text{Ker}(ad^{k_i}(e_0))$ ,
- (c) for  $i \neq j$ ,  $h_i \perp h_j$ .

*Remark 3.3.* If an element  $h \in \mathfrak{h}$  satisfies  $ad^{k_i+1}(e_0)(h) = 0$ , then  $h \in (V_1 \oplus \dots \oplus V_i) \cap \mathfrak{h}$ .

We will end this section by giving two examples of principal basis.

*Example 3.4.* Let  $\mathfrak{g}$  be the Lie algebra of type  $G_2$ . Its Cartan subalgebra  $\mathfrak{h}$  is 2-dimensional. Let  $\alpha_1$  and  $\alpha_2$  be a choice of the simples roots. We choose  $B$  to be the canonical form, i.e. on simple root it is given by

$$B(\alpha_1, \alpha_1) = 2 \quad B(\alpha_1, \alpha_2) = -1 \quad B(\alpha_2, \alpha_2) = \frac{2}{3}.$$

The coroots are then given by

$$\alpha_1^\vee = B^b(\alpha_1) \quad \alpha_2^\vee = 3B^b(\alpha_2).$$

The first vector in the principal basis is the sum of the positive coroots, and the second is one orthogonal to the first. They are (up to constant)

$$h_1 = 5\alpha_1^\vee + 3\alpha_2^\vee \quad h_2 = -3\alpha_1^\vee + \alpha_2^\vee.$$

*Example 3.5* (Principal basis for  $\mathfrak{sl}(3, \mathbb{C})$ ). Let  $e_{ij}$  be the canonical basis of  $\text{Mat}(3, \mathbb{C})$ . The standard choice of basis for the Cartan subalgebra of  $\mathfrak{sl}(3, \mathbb{C})$  is  $\tilde{h}_1 = e_{11} - e_{22}$  and  $\tilde{h}_2 = e_{22} - e_{33}$ . The positive (resp. negative) nilpotent elements are given by  $e_{12}$ ,  $e_{23}$  and  $e_{13}$  (resp.  $e_{21}$ ,  $e_{32}$  and  $e_{31}$ ). In terms of this basis the principal  $sl_2$ -triple  $\mathfrak{s}_0$  is given by  $h_0 = 2\tilde{h}_1 + 2\tilde{h}_2$ ,

$e_0 = e_{12} + e_{23}$  and  $f_0 = 2e_{21} + 2e_{32}$ . In the decomposition (5) the two irreducible  $\mathfrak{s}_0$ -modules are given by

$$V_1 = \langle e_0, h_0, f_0 \rangle_{\mathbf{C}},$$

$$V_2 = \langle e_{13}, e_{23} - e_{12}, \tilde{h}_1 - \tilde{h}_2, e_{32} - e_{21}, e_{31} \rangle_{\mathbf{C}}.$$

Then the principal basis is given by (up to constants)

$$\{h_1 = \tilde{h}_1 + \tilde{h}_2 = e_{11} - e_{33}, h_2 = \tilde{h}_1 - \tilde{h}_2 = e_{11} - 2e_{22} + e_{33}\}.$$

By a direct calculation we verify that it is an orthogonal basis.

#### 4. HOW TO OBTAIN THE PRINCIPAL BASIS OF THE CARTAN SUBALGEBRA USING THE GENERATORS OF THE INVARIANTS POLYNOMIALS

Let  $\mathfrak{g}$  be a simple Lie algebra over the complex field, and  $\tilde{p} \in S(\mathfrak{g}^*)^{\mathfrak{g}}$  an invariant polynomial. We consider  $d\tilde{p}$  the differential of this polynomial. It is defined by

$$d\tilde{p} = \sum_i \frac{\partial \tilde{p}}{\partial e_i^*} \otimes e_i^* \in S(\mathfrak{g}^*) \otimes \mathfrak{g}^*,$$

where  $\{e_i^*\}$  is an orthonormal basis of  $\mathfrak{g}^*$ . We view this differential as an map from  $\mathfrak{g}$  to  $\mathfrak{g}^*$ , i.e.

$$\mathfrak{g} \ni x \longrightarrow d\tilde{p}(x) = \sum_i \frac{\partial \tilde{p}}{\partial e_i^*}(x) \cdot e_i^* \in \mathfrak{g}^*.$$

Let  $G = \text{Int}(\mathfrak{g})$  be the group of inner automorphisms of  $\mathfrak{g}$ . It is generated by the elements of the form  $e^{\text{ad}_x}$  with  $x \in \mathfrak{g}$  nilpotent. The invariant bilinear form  $B$  is invariant by  $G$ , i.e. for all  $g \in G$  and  $x, y \in \mathfrak{g}$  we have  $B(g \cdot x, y) = B(x, g^{-1} \cdot y)$ . Moreover,  $G$  intertwines the isomorphisms  $B^b$  and  $B^{\natural}$ .

**Proposition 4.1.** *The map  $d\tilde{p}$  is  $G$ -equivariant, i.e. for all  $g \in G$*

$$d\tilde{p} \circ g = g \circ d\tilde{p}.$$

*Proof.* Let  $x \in \mathfrak{g}$  and  $g \in G$ , then  $d\tilde{p}(g \cdot x) = \sum_i \frac{\partial \tilde{p}}{\partial e_i^*}(g \cdot x) e_i^*$ . Let  $y_i = g^{-1} x_i$  be a new orthonormal basis of  $\mathfrak{g}$ . Then using the fact that  $p$  is  $G$ -invariant we obtain

$$\sum_i \frac{\partial \tilde{p}}{\partial e_i^*}(g \cdot x) e_i^* = \sum_i \frac{\partial \tilde{p}}{\partial (g \cdot y_i)^*}(g \cdot x) g \cdot y_i = \sum_i \frac{\partial (g \cdot \tilde{p})}{\partial (g \cdot y_i)^*}(g \cdot x) g \cdot y_i = \sum_i \frac{\partial \tilde{p}}{\partial y_i^*}(x) g \cdot y_i.$$

□

Let  $D\tilde{p}$  be the composition of  $d\tilde{p}$  with  $B^b$ , i.e.

$$D\tilde{p} = B^b \circ d\tilde{p} : \mathfrak{g} \longrightarrow \mathfrak{g}.$$

Obviously  $D\tilde{p}$  is  $G$ -equivariant.

We can consider the same construction with an invariant polynomial  $p \in S(\mathfrak{h}^*)^{\mathcal{W}}$ . Then we obtain the map

$$Dp : \mathfrak{h} \longrightarrow \mathfrak{h}.$$

The following proposition relates these two constructions.

**Proposition 4.2.** *Let  $p \in S(\mathfrak{h}^*)^{\mathcal{W}}$  and  $\tilde{p} \in S(\mathfrak{g}^*)^{\mathfrak{g}}$  be invariant polynomials such that  $\text{Chev}(\tilde{p}) = p$ . Then for all  $h \in \mathfrak{h}$  we have*

$$Dp(h) = D\tilde{p}(h).$$

*Proof.* By Lemma 7.3.6 of [Dix], there exist  $a \in S(\mathfrak{g}^*)^{\mathfrak{g}} \mathfrak{n}_+^*$  such that  $\tilde{p} = p + a$ . But  $Da(h) = 0$  because  $\mathfrak{h}^*$  is orthogonal to  $\mathfrak{n}_+^*$ .  $\square$

Let  $\{\tilde{p}_1, \dots, \tilde{p}_r\}$  be a choice of generators of  $S(\mathfrak{g}^*)^{\mathfrak{g}}$ , with  $\deg(p_1) \leq \deg(p_2) \leq \dots \leq \deg(p_r)$ . Let  $\rho^\vee \in \mathfrak{h}$  be the half sum of positive coroots. By the preceding proposition the elements

$$(8) \quad \{D\tilde{p}_1(\rho^\vee), \dots, D\tilde{p}_r(\rho^\vee)\}$$

are in  $\mathfrak{h}$ . Note that up to a constant we can chose  $\tilde{p}_1$  such that  $D\tilde{p}_1(\rho^\vee) = \rho^\vee$ . By Theorem 3 of [Var] with  $A = \rho^\vee$ , this family is linearly independent. Hence it is a basis of  $\mathfrak{h}$ . Using the Gram-Schmidt orthogonalization process on this basis we obtain the orthogonal basis :

$$(9) \quad \begin{array}{c} \{D\tilde{p}_1(\rho^\vee), \dots, D\tilde{p}_r(\rho^\vee)\} \\ \downarrow \text{Gram Schmidt} \end{array}$$

$$\begin{aligned} \{h_1 &= D\tilde{p}_1(\rho^\vee), \\ h_2 &= D\tilde{p}_2(\rho^\vee) - \lambda_{21}D\tilde{p}_1(\rho^\vee), \\ &\dots, \\ h_r &= D\tilde{p}_r(\rho^\vee) - \lambda_{r,r-1}D\tilde{p}_{r-1}(\rho^\vee) - \dots\}. \end{aligned}$$

The first result of this section is the following Theorem.

**Theorem 4.3.** *The orthogonal basis (9) of  $\mathfrak{h}$  is (up to constant multiple) the principal basis of the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ .*

Before we prove this Theorem we will show how it work in the Examples 3.4 and 3.5.

*Example 4.4* (Continuation of Example 3.4). We will compute the principal basis using Theorem 4.3 and then compare with the previous result. Let  $\{x, y\}$  be the orthonormal basis of the Cartan subalgebra given by

$$x = \frac{\alpha_1^\vee}{\sqrt{2}} \quad y = \sqrt{\frac{2}{3}} \left( \alpha_2^\vee + \frac{3}{2}\alpha_1^\vee \right).$$

Let  $\{x^*, y^*\}$  be the dual basis. A choice for generators of  $S(\mathfrak{h}^*)^{\mathcal{W}}$  is given by

$$p_1 = (x^*)^2 + (y^*)^2 \quad p_2 = 33(x^*)^6 + 27(y^*)^6 + 45(x^*)^4(y^*)^2 + 135(x^*)^2(y^*)^4.$$



The differentials of these polynomials computed at  $\rho^\vee$  give (up to constant multiple)

$$Dp_1\rho^\vee = \text{cte}\rho^\vee \quad Dp_2\rho^\vee = \text{cte}2425\alpha_1^\vee + 1383\alpha_2^\vee.$$

We remark that they are not orthogonal, but after orthogonalization we obtain the principal basis.

*Remark 4.5.* The previous example shows that the orthogonalization is indispensable.

*Example 4.6* (Continuation of example 3.5). We will compute the principal basis using Theorem 4.3 and then compare with the previous result. A choice of generators of  $S(\mathfrak{h}^*)^\mathcal{W}$  is given by ([Me], Section 3),

$$p_1 = (e_{11}^*)^2 + (e_{22}^*)^2 + (e_{33}^*)^2 \quad p_2 = (e_{11}^*)^3 + (e_{22}^*)^3 + (e_{33}^*)^3,$$

where  $\{e_{11}^*, e_{22}^*, e_{33}^*\}$  is the dual basis of  $\{e_{11}, e_{22}, e_{33}\}$ . As explain in [Me], these polynomials belong to the dual of the Cartan subalgebra of  $gl(3, \mathbb{C})$ . We work with the basis  $\{\tilde{h}_1^* = e_{11}^* - e_{22}^*, \tilde{h}_2^* = e_{22}^* - e_{33}^*\}$  completed by the central element  $c^* = e_{11}^* + e_{22}^* + e_{33}^*$ . In this basis these polynomials are written as follows:

$$p_1 = (\tilde{h}_1^*)^2 + (\tilde{h}_2^*)^2 + \tilde{h}_1^*\tilde{h}_2^* + (c^*)^2 \quad p_2 = 2(\tilde{h}_1^*)^3 - 2(\tilde{h}_2^*)^3 + 3(\tilde{h}_1^*)^2\tilde{h}_2^* - 3\tilde{h}_1^*(\tilde{h}_2^*)^2 + \text{terms with } c^*.$$

Remark that  $\langle \tilde{h}_1^*, \tilde{h}_2^* \rangle_{\mathbb{C}} \perp \langle c^* \rangle_{\mathbb{C}}$ . Their differentials evaluated at  $\rho^\vee$  are (up to a constant)

$$Dp_1(\rho^\vee) = \tilde{h}_1 + \tilde{h}_2 \quad Dp_2(\rho^\vee) = \tilde{h}_1 - \tilde{h}_2.$$

This is fortunately the same basis as before. Note that in this case the choice of polynomial is unique (up to constants) and we do not need to orthogonalize.

*Proof of Theorem 4.3.* This proof work if all exponents are different. This is the case for all simple Lie algebras except for type  $D_l$  with  $l$  even. We proof this case in §5.

- (a) Using Theorem 3 of [Var] with  $A = \rho^\vee$ , we conclude that the elements (8) are linearly independents.
- (b) Let  $\mathfrak{s}_0$  be the principal  $sl_2$ -triple given by (6). Then  $D\tilde{p}_i(\rho^\vee) \in \text{Ker}(\text{ad}_{e_0}^{k_i+1})$ . Indeed using the  $G$ -equivariance of  $D\tilde{p}_i$  (Proposition 4.1) for  $e^{\text{tad}_{e_0}}$  ( $t$  a parameter) we have

$$(e^{\text{tad}_{e_0}} D\tilde{p}_i)(\rho^\vee) = D\tilde{p}_i(e^{\text{tad}_{e_0}} \rho^\vee) = (D\tilde{p}_i \circ (1 + \text{tad}_{e_0}))(\rho^\vee),$$

but and  $D\tilde{p}_i$  is a polynomial of degree  $k_i$ , and the term in  $t^{k_i+1}$  is vanishes.

- (c) The step (b) implies that for all  $i$

$$D\tilde{p}_i(\rho^\vee) \in (V_1 \oplus \cdots \oplus V_i) \cap \mathfrak{h}.$$

- (d) Then if we orthogonalize the family

$$\{D\tilde{p}_1(\rho^\vee), \dots, D\tilde{p}_r(\rho^\vee)\}$$

as (9) we obtain (up to constant multiple) the principal basis of  $\mathfrak{h}$ .

□

This Theorem provides a method to compute the principal basis of the Cartan subalgebra of  $\mathfrak{h}$  using a set of generators of the invariants polynomials  $S(\mathfrak{g}^*)^\mathfrak{g}$ . Now we will give the second result of this section, which explain how to obtain the principal basis of the Cartan subalgebra of

$\mathfrak{g}^\vee$  the Langlands dual of  $\mathfrak{g}$  using a set of generators of  $S(\mathfrak{g}^*)^\mathfrak{g}$ .

As explained before  $d\tilde{p}$  is an  $G$ -equivariant map from  $\mathfrak{g}$  to  $\mathfrak{g}^*$ . We consider the following map, which is also  $G$ -equivariant,

$$\hat{D}\tilde{p} = d\tilde{p} \circ B^\natural : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*.$$

We have the analogue of Proposition 4.2

**Proposition 4.7.** *Let  $p \in S(\mathfrak{h}^*)^\mathcal{W}$  and  $\tilde{p} \in S(\mathfrak{g}^*)^\mathfrak{g}$  be invariant polynomials such that  $\text{Chev}(\tilde{p}) = p$ . Then for all  $\lambda \in \mathfrak{h}^*$  we have*

$$\hat{D}p(\lambda) = \hat{D}\tilde{p}(\lambda).$$

The proof is the same as for Proposition 4.2.

The second result of this section is the following theorem.

**Theorem 4.8.** *Let  $\rho$  be the half sum of the positive roots of  $\mathfrak{g}$ . Let  $\{\hat{D}\tilde{p}_1(\rho), \dots, \hat{D}\tilde{p}_r(\rho)\}$  be a family of element of  $\mathfrak{h}^*$ , the Cartan subalgebra of  $\mathfrak{g}^\vee$ . Then*

- (a) *This family is linearly independent.*
- (b) *Using the Gram-Schmidt process as in (9) we obtain (up to constant) the principal basis of  $\mathfrak{h}^\vee \subset \mathfrak{g}^\vee$ .*

*Proof.* (a) Let  $p \in S(\mathfrak{h}^*)^\mathcal{W}$  be an invariant polynomial and define  $p^b = B^b(p) \in S((\mathfrak{h}^\vee)^*)^\mathcal{W}^\vee$ . As for  $p$ , we consider  $dp^b$  to be a map from  $\mathfrak{h}^*$  to  $\mathfrak{h}$ . We have the following relation between  $dp$  and  $dp^b$ ,

$$dp \circ B^b = B^\natural \circ dp^b.$$

- (b) We take  $\tilde{p} \in S(\mathfrak{g}^*)^\mathfrak{g}$  such that  $\text{Chev}(\tilde{p}) = p$ . Using the Proposition 4.7 and (a) we have for all  $\lambda \in \mathfrak{h}^*$ ,

$$\hat{D}\tilde{p}(\lambda) = \hat{D}p(\lambda) = (B^\natural \circ dp^b)(\lambda).$$

- (c) We take  $\hat{p} \in S((\mathfrak{g}^\vee)^*)^\mathfrak{g}^\vee$  such that  $\text{Chev}(\hat{p}) = p^b$ . We use the equalities in (b) to obtain

$$\hat{D}\tilde{p}(\lambda) = \hat{D}p(\lambda) = (B^\natural \circ dp^b)(\lambda) = (B^\natural \circ d\hat{p})(\lambda).$$

The proof of the last equality is the same as for the Proposition 4.2 with  $\mathfrak{g}^\vee$  at the place of  $\mathfrak{g}$ .

- (d) By (c) point we see that the family  $\{\hat{D}\tilde{p}_1(\rho), \dots, \hat{D}\tilde{p}_r(\rho)\}$  is the same as the family  $\{(B^\natural \circ d\hat{p}_1^b)(\rho), \dots, (B^\natural \circ d\hat{p}_r^b)(\rho)\}$ . The result (a) is given by Theorem 3 in [Var] and the result (b) by using Theorem 4.3 with  $\mathfrak{g}^\vee$  replacing  $\mathfrak{g}$ .

□

**Remark 4.9** (The Kostant conjecture about the principal basis). Let  $\text{Cl}(\mathfrak{g}, K)$  be the Clifford algebra of the Lie algebra  $\mathfrak{g}$  and  $K$  its Killing form. Consider the Harish-Chandra projection  $\Phi : \text{Cl}(\mathfrak{g})^\mathfrak{g} \rightarrow \text{Cl}(\mathfrak{h})$  defined by the decomposition  $\text{Cl}(\mathfrak{g})^\mathfrak{g} = \text{Cl}(\mathfrak{h}) \oplus \text{Cl}(\mathfrak{g})\mathfrak{n}_+ \cap \text{Cl}(\mathfrak{g})^\mathfrak{g}$ , where  $\mathfrak{n}_+$

are the positives nilpotent elements. In particular Kostant showed that the image of the Harish-Chandra projection of the primitive generators of  $\text{Cl}(\mathfrak{g})^{\mathfrak{g}} \cong \bigwedge \mathfrak{g}^{\mathfrak{g}}$  are contained in  $\mathfrak{h} \subset \text{Cl}(\mathfrak{h})$ . Kostant conjectured that the Harish-Chandra projection of the primitive generators of  $\mathfrak{g}$  gives the principal basis of the Cartan Subalgebra of the Langlands dual  $\mathfrak{g}^{\vee}$  (here using  $K$ ,  $\mathfrak{h}$  and  $\mathfrak{h}^{\vee}$  are identified). This conjecture was partially proven by Bazlov in his thesis [Baz].

### 5. THE $so(2l, \mathbb{C})$ CASE ( $l$ EVEN)

Throughout this section  $\mathfrak{g} = so(2l, \mathbb{C})$  with  $l$  even,  $\mathfrak{h} \subset \mathfrak{g}$  is a Cartan subalgebra,  $\mathcal{W}$  the Weyl group and  $B$  is an invariant bilinear form. The root system of  $\mathfrak{g}$  is  $D_l$  and the exponents are given by [Bou], Table IV

$$\{1, 3, 5, \dots, l-1, l-1, \dots, 2l-5, 2l-3\},$$

Then in the decomposition of  $\mathfrak{g}$  into irreducible  $\mathfrak{s}_0$ -modules,

$$\mathfrak{g} = V_1 \oplus \dots \oplus V_{l/2} \oplus V_{l/2+1} \oplus \dots \oplus V_l,$$

the modules  $V_{l/2}$  and  $V_{l/2+1}$  have the same dimension  $2l-1$ .

This decomposition does not determine  $V_{l/2}$  and  $V_{l/2+1}$  uniquely. Indeed  $V_{l/2}$  and  $V_{l/2+1}$  are generated by the highest weight vectors  $v_1$  and  $v_2$ , i.e.  $V_{l/2} = U(\mathfrak{s}_o)v_1$  and  $V_{l/2+1} = U(\mathfrak{s}_o)v_2$ . But  $v_1 + v_2$  and  $v_1 - v_2$  also generate two irreducible  $\mathfrak{s}_o$ -modules whose the direct sum is equal to  $V_{l/2} \oplus V_{l/2+1}$ , i.e.  $U(\mathfrak{s}_o)(v_1 + v_2) \oplus U(\mathfrak{s}_o)(v_1 - v_2) = V_{l/2} \oplus V_{l/2+1}$ .

The following decomposition is uniquely determined by the action of the principal  $sl_2$ -triple,

$$(10) \quad \mathfrak{g} = V_1 \oplus \dots \oplus V_{l/2-1} \oplus W \oplus V_{l/2+2} \oplus \dots \oplus V_l,$$

where  $W = V_{l/2} \oplus V_{l/2+1}$ . Note that this decomposition is orthogonal relatively to  $B$ .

Hence the principal basis is uniquely determined modulo a choice of two linearly independent vectors in  $W \cap \mathfrak{h}$ , i.e.

$$\text{principal basis of } \mathfrak{h} = \{h_1, \dots, h_l, h_{l+1}, \dots, h_r\},$$

with  $h_i$  such that  $\langle h_i \rangle_{\mathbb{C}} = V_i \cap \mathfrak{h}$  for all  $i \neq l/2, l/2+1$  and  $\langle h_{l/2}, h_{l/2+1} \rangle_{\mathbb{C}} = W \cap \mathfrak{h}$ . Moreover if we chose  $h_{l/2}$  and  $h_{l/2+1}$  to be orthogonal, then the principal basis is uniquely determined modulo a rotation in  $W$ .

Now we will give the modifications in the proof of Theorem 4.3.

(a) (b) The first and second step remain true.

(c) The third step is modified as follows: for all  $i \neq l/2, l/2+1$  we have

$$D\tilde{p}_i(\rho^{\vee}) \in (V_1 \oplus \dots \oplus V_i) \cap \mathfrak{h},$$

and for  $i = l/2$  or  $i = l/2+1$  we have

$$D\tilde{p}_i(\rho^{\vee}) \in (V_1 \oplus \dots \oplus W) \cap \mathfrak{h}.$$

(d) The fourth step remaining true with a suitable choice of basis vectors in  $W \cap \mathfrak{h}$ .

Let  $\{\alpha_1, \dots, \alpha_l\}$  be the  $l$  simple roots of  $\mathfrak{g}$ . We define the automorphism  $\sigma$  by

$$\sigma(\alpha_i) = \alpha_i \quad (i = 1, \dots, l-2) \quad \sigma(\alpha_{l-1}) = \sigma(\alpha_l) \quad \sigma(\alpha_l) = \sigma(\alpha_{l-1}).$$

This automorphism induces an automorphism on the Lie algebra, see Section 7.9 of [Kac]. As this automorphism is of order two it is diagonalizable and its eigenvalues are  $+1$  and  $-1$ . Let  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  be the decomposition in the eigenspace  $\mathfrak{g}_0$ , resp.  $\mathfrak{g}_1$ , of eigenvalues  $+1$ , resp.  $-1$ . In [Kac], Chapter 8, Kac showed that the roots system of  $\mathfrak{g}_0$  is  $B_{l-1}$ . This implies that the dimension of  $\mathfrak{g}_1$  is  $2l - 1$ . Indeed  $\dim(D_l) = l(2l - 1)$  and  $\dim(B_{l-1}) = l(2l - 1)$ .

We denote by  $B$  the canonical form of  $\mathfrak{g}$ , i.e.  $B(\alpha_i, \alpha_i) = 2$ .

**Theorem 5.1.** *Consider the decomposition (10) of  $\mathfrak{g}$ , we have*

- (a) *All the irreducible  $\mathfrak{s}_0$ -modules  $V_i$  are contained in the eigenspace  $\mathfrak{g}_0$ .*
- (b) *The  $\mathfrak{s}_0$ -module  $W$  is invariant by  $\sigma$ , i.e.  $\sigma(W) = W$ .*
- (c) *The automorphism  $\sigma$  provides the decomposition of  $W$  into  $W = W_0 \oplus W_1$ , where  $W_i \subset \mathfrak{g}_i$ . Moreover, this is a decomposition into two irreducible  $\mathfrak{s}_0$ -submodules and they are orthogonal.*

*Hence if we require that the elements of the principal basis should be eigenvectors of  $\sigma$ , then the principal basis is uniquely determined (up to constants).*

The proof of this Theorem follows from the next three propositions.

**Proposition 5.2.** *Let  $\mathfrak{s}_0 = \langle e_0, f_0, h_0 \rangle$  be the principal  $sl_2$ -triple. Then its adjoint action commute with the automorphism  $\sigma$ , i.e.*

$$\sigma \circ ad_{e_0} = ad_{e_0} \circ \sigma \quad \sigma \circ ad_{f_0} = ad_{f_0} \circ \sigma \quad \sigma \circ ad_{h_0} = ad_{h_0} \circ \sigma.$$

*Proof.* The principal TDS  $\mathfrak{s}_0$  is given by

$$h_0 = 2(l-1)\alpha_1^\vee + 2(2l-3)\alpha_2^\vee + \dots + (l-2)(l+1)\alpha_{l-2}^\vee + l(l-1)/2(\alpha_{l-1}^\vee + \alpha_l^\vee),$$

$$e_0 = e_1 + \dots e_l,$$

$$f_0 = 2(l-1)f_1 + \dots + (l-2)(l+1)f_{l-2} + l(l-1)/2(f_{l-1} + f_l),$$

where  $e_i$  is the positive nilpotent element corresponding to the simple root  $\alpha_i$ , and the  $f_i$  is the corresponding negative nilpotent elements such that  $B(e_i, f_i) = 1$ . Clearly they are eigenvectors of eigenvalues  $+1$ . The result follows from the equation

$$\sigma(ad_{e_0}x) = ad_{\sigma(e_0)}\sigma(x) = ad_{e_0}\sigma(x).$$

The proof for  $h_0$  and  $f_0$  is similar. □

**Proposition 5.3.** *Let  $V \in \mathfrak{g}$  be a irreducible  $\mathfrak{s}_0$ -module. If there exist  $x \in V$  such that  $\sigma(x) = x$  (or  $\sigma(x) = -x$ ), then  $V \subset \mathfrak{g}_0$  (or  $V \subset \mathfrak{g}_1$ ).*

*Proof.* Let  $x \in V$  be a eigenvector of eigenvalue  $\lambda$  for  $\sigma$  ( $\lambda = \pm 1$ ). Then using Proposition 5.2, we have that all vectors  $y \in V = U(\mathfrak{s}_0)x$  are eigenvectors of eigenvalue  $\lambda$ . □

Before the next proposition we introduce the standard choice for the generators of  $S(\mathfrak{h}^*)^{\mathcal{W}}$ .

Let  $\{e_1^*, \dots, e_l^*\}$  be the standard orthonormal basis of  $\mathfrak{h}^*$ . The simple roots are given by

$$\alpha_1 = e_1^* - e_2^* \quad \cdots \quad \alpha_{l-1} = e_{l-1}^* - e_l^* \quad \alpha_l = e_l^* + e_{l-1}^*.$$

Recall that on the elements of the basis the canonical bilinear invariant form is given by  $B(e_i^*, e_j^*) = \delta_{ij}$ . In term of this basis the standard generators are given by ([Me], Section 3)

$$p_i = \sum_j (e_j^*)^{2i} \quad i = 1, \dots, l-1 \quad \text{and} \quad p_e = e_1^* e_2^* \dots e_l^*.$$

The half sum of positive roots is given by

$$\rho^\vee = (l-1)(e_1) + (l-2)(e_2) + \dots + e_{l-1},$$

where  $\{e_1, \dots, e_l\}$  is the dual basis of  $\{e_1^*, \dots, e_l^*\}$ . Note that the two polynomials of degree  $l$  are  $p_e$  and  $p_{l/2}$ .

The differential of these generators computed at  $\rho^\vee$  are,

$$\begin{aligned} Dp_1(\rho^\vee) &= 2\rho^\vee \\ Dp_2(\rho^\vee) &= 4 \cdot 2^3((l-1)^3 e_1 + \dots + e_{l-1}) \\ &\dots \\ Dp_e(\rho^\vee) &= l! \cdot 2^{l-1} e_l \\ Dp_{l/2}(\rho^\vee) &= \dots \\ &\dots \\ Dp_{l-1}(\rho^\vee) &= (2l-2)2^{2l-3}((l-1)^{2l-3} e_1 + \dots + e_{l-1}) \end{aligned}$$

We remark that  $\langle Dp_1(\rho^\vee), \dots, Dp_{l-1}(\rho^\vee) \rangle_{\mathbb{C}}$  is orthogonal to  $\langle Dp_e(\rho^\vee) \rangle_{\mathbb{C}}$ .

From these calculations we deduce the next proposition

**Proposition 5.4.** *We have that  $Dp_i(\rho^\vee) \in \mathfrak{g}_0$  for all  $i = 1, \dots, l-1$  and  $Dp_e(\rho^\vee) \in \mathfrak{g}_1$ .*

Now we prove the Theorem 5.1.

*Proof of Theorem 5.1.* (a) We orthogonalize the basis (11) and obtain the principal basis  $\{h_1, \dots, h_l\}$ . Note that all  $h_i$  except  $h_{l/2}$  are in the space  $\langle e_1, \dots, e_{l-1} \rangle_{\mathbb{C}}$  and  $h_{l/2} \in \langle e_l \rangle_{\mathbb{C}}$ . By Proposition 5.3, this proves the first assertion.

(b) The second assertion follows from Proposition 5.3 and the fact that  $W = U(\mathfrak{s}_0) \langle h_{l/2}, h_{l/2+1} \rangle_{\mathbb{C}}$ .

(c) For the third assertion we have that  $W_0 = U(\mathfrak{s}_0)h_{l/2+1}$  and  $W_1 = U(\mathfrak{s}_0)h_{l/2}$ .

□

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